LIE ALGEBRA CONFIGURATION PAIRING

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Abstract. We give a new description of the configuration pairing of [13] by showing that it computes coefficients in the associative, preLie, or graph polynomial of a Lie bracket expression. We also connect the graph complexes of [13] with preLie algebras. Examples apply the combinatorial definition of the configuration pairing to compute coefficients in the associative (noncommutative) polynomial of Lie elements. This outlines a new way of understanding and computing with Lie algebras.

1. Introduction

The configuration pairing of graphs and trees has its genesis in [11] as an explicit geometric description of the homology/cohomology pairing for configuration spaces. Cycles in the configuration space homology are realized by [11] as submanifolds, where individual points orbit as systems and galaxies. Cohomology cocycles check whether certain arrangements of points can ever occur in a homology galaxy. Algebraically, homology galaxies are written as trees and cohomology cocycles are written as directed graphs. Antisymmetry and Jacobi expressions of trees bound, and so vanish in homology; also arrow-reversing and Arnold expressions of graphs cobound. The homology of configuration spaces is the Poisson operad, so this gives a model for the Poisson cooperad and an equivalence of Lie coalgebras and directed graphs modulo arrow-reversing and Arnold. This is exploited in [13], [14], and [15].

In this paper we give an alternate view of the configuration pairing, grounded not in topology but algebra. Standard operad maps induce a commutative diagram of functors of categories.

(1)
$$\begin{tikzcd}
 & \text{Graph algebras} \arrow[dr, two heads] & & \\
 & & \text{preLie algebras} \arrow[dr, two heads] & \\
\text{Lie algebras} \arrow[uur] \arrow[urr, tail] \arrow[rrr, tail] & & & \text{Associative algebras}
\end{tikzcd}$$

The map from Lie algebras to associative algebras is the universal enveloping algebra map. By analogy, we call the other maps from Lie algebras also "universal enveloping" maps – the map to preLie algebras is standard and we will describe the map to graph algebras later. The maps marked $\twoheadrightarrow$ are quotient maps on algebras. Up to slight tweaks to coalgebra structure, we construct a dual diagram in coalgebras.

The dual maps to Lie coalgebras yield presentations of Lie coalgebras as quotients of graph, preLie, and associative coalgebras. The duality of Lie algebras with the graph presentation of Lie coalgebras is the combinatorial pairing of [13]. The duality with the associative coalgebra presentation is given by computing coefficients in the associative polynomial of a Lie expression. The preLie presentation interpolates between these two.

Section 2 recalls the classical situation of Lie algebras and universal enveloping associative algebras. We arrange ideas and notation anticipating our later sections. In the context of associative algebras, the configuration pairing of a word and Lie bracket expression is the coefficient of the word in the associative (noncommutative) polynomial of the Lie bracket expression. Material in this section is all classical aside possibly from the presence of Lie and associative coalgebra structures in Proposition 2.4 and Corollary 2.8.

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In Section 3 we work analogously to Section 2, but with preLie structures. We spend more time on preliminaries since we expect this to be less familiar territory. Our construction of Lie coalgebra structures from preLie structures is similar in motivation to [6], though we work with duals and use the configuration pairing. In this setting we develop two views of the configuration pairing. The algebraic configuration pairing is defined similar to Section 2, as reading the coefficient of a preLie element in the preLie polynomial of a Lie bracket expression. The combinatorial configuration pairing is defined in terms of vertex-labeled rooted preLie trees and leaf-labeled Lie trees in the spirit of [13].

Section 4 is motivated by the presentation of Lie coalgebras via preLie coalgebras given at the end of Section 3. This presentation is simplified by replacing preLie algebras by graph algebras. A theory analogous to the previous sections holds for graph algebras. Proofs in this realm are very simple. Furthermore the structures presented in Sections 2 and 3 are all induced by graph algebra structures via quotient maps to preLie algebras and associative algebras.

In the appendix we give operad-level constructions and proofs. We also give a description of the full graph algebra and coalgebra structure, which we omit from Section 4 for simplicity.

Throughout, we will assume that our algebras have underlying k-vector spaces. In particular, we make frequent use of the free algebra maps from k-vector spaces to algebras. For brevity, we write $\otimes$ for $\otimes_k$. In remarks, we discus interpretations of definitions and propositions, given a chosen basis $B = \{b_i\}_{i\in I}$ of a k-vector space V.

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2. The Configuration Pairing with Associative Coalgebras

We recall the classical theory of Lie algebras and their universal enveloping algebras, setting notation for later sections, and carefully developing the linear dual of the universal enveloping algebra map (to avoid concerns about infinite dimensional coalgebra structures).

Given a k-vector space V, $\mathbb{T}V = \bigoplus_{n\geq 1} V^{\otimes n}$ is the free nonunital associative algebra on V. Write the word $x_1 \cdots x_n$ for the element $x_1 \otimes \cdots \otimes x_n \in \mathbb{T}V$. The universal enveloping associative algebra of a Lie algebra is $U_A L = \mathbb{T}L/\sim$, where $[x, y] \sim xy - yx$. The associative (noncommutative) polynomial of a Lie element is given by $p_A : L \to U_A L$, the composition $L \hookrightarrow \mathbb{T}L \twoheadrightarrow U_A L$. Write $\mathbb{L}V$ for the free Lie algebra on V and recall the classical isomorphism $U_A\mathbb{L}V \cong \mathbb{T}V$. We are interested in the map $p_A : \mathbb{L}V \to \mathbb{T}V$ and its dual. Let $(\mathbb{T}V)^*$ and $(\mathbb{L}V)^*$ be the vector space duals of $\mathbb{T}V$ and $\mathbb{L}V$.

Remark 2.1. In order to have honest coalgebra structures on duals, some type of finiteness condition must be enforced. If V is finite dimensional, then we may use the length filtration of $\mathbb{T}V$, requiring $\psi \in (\mathbb{T}V)^*$ to satisfy $\psi(w) = 0$ for all but finitely many lengths of w. If V is not finite dimensional, then we may choose a basis B, use this to give a basis of $\mathbb{T}V$, and impose a finiteness condition on ψ via this basis. Alternately we may weaken the definition of coalgebra to allow formal sums in the coproduct operation. The coproduct operation then lands in the completed tensor product $\Delta : C \to C \mathbin{\widehat{\otimes}} C$ and the coalgebra axioms are all modified accordingly.

Throughout this paper, all constructions will be grounded via pairings with algebras. Due to finiteness conditions on the algebra side, these pairings will never involve infinite sums of nonzero elements, no matter which of the above alternatives for a specific definition of dual is adopted. Since our goal is an understanding of algebras via our explicit pairings, we are ambivalent about issues related to infinite dimensional coalgebras.

Definition 2.2. Define the vector space pairing $\langle -, - \rangle : (\mathbb{T}V)^* \otimes \mathbb{L}V \to k$ to be $\langle \psi, \ell \rangle = \psi\big(p_A(\ell)\big)$.

Let $\eta_A : (\mathbb{T}V)^* \to (\mathbb{L}V)^*$ be the associated map $\psi \mapsto \langle \psi, - \rangle$.

Remark 2.3. The map η_A is the dual of p_A as a map of vector spaces.

If V has a chosen basis B, then we may canonically write $p_A(\ell)$ as a polynomial of words in the alphabet B. Furthermore, elements of $(\mathbb{T}V)^*$ are uniquely formal sums of w^* where $w^*(u) = \delta(w, u)$ for w, u words in B. In this case, $\langle w^*, \ell \rangle = w^*(p_A(\ell))$ is the coefficient of the word w in the associative polynomial $p_A(\ell)$.

The induced coalgebra structure on $(\mathbb{T}V)^*$ cuts a word at all possible positions $\Delta w^* = \sum_{w=ab} a^* \otimes b^*$. Due to the definition of p_A, the map η_A will be a map of coalgebras only after twisting the coalgebra structure of $(\mathbb{T}V)^*$ to be anti-commutative. Define the cobracket to be $]\psi[= \Delta\psi - \tau\Delta\psi$ where τ is the twist map. On homogeneous elements the cobracket is $]w^*[= \sum_{w=ab} (a^* \otimes b^* - b^* \otimes a^*)$.

Proposition 2.4. *If $\psi \in (\mathbb{T}V)^*$ then $\big\langle \psi, [x, y] \big\rangle = \sum_i \langle \alpha_i, x \rangle \langle \beta_i, y \rangle$, where $]\psi[= \sum_i \alpha_i \otimes \beta_i$.*

Proof. This follows immediately from $p_A([\ell_1, \ell_2]) = p_A(\ell_1)p_A(\ell_2) - p_A(\ell_2)p_A(\ell_1)$ for homogeneous Lie bracket expressions ℓ_1 and ℓ_2. □

By Proposition 2.4 $\langle -, - \rangle$ is a coalgebra/algebra pairing if $(\mathbb{T}V)^*$ is given the cobracket coalgebra structure. From now on, we will always equip $(\mathbb{T}V)^*$ with the cobracket coalgebra structure.

Example 2.5. Proposition 2.4 gives a method for recursive calculation of coefficients in $p_A(\ell)$. For example the coefficient of $abbba$ in $p_A([[[b, a], b], [a, b]])$ is the following.

$$\begin{aligned}
\langle abbba^*, [[[b,a],b],[a,b]] \rangle &= \langle abb^*, [[b,a],b] \rangle \langle ba^*, [a,b] \rangle \; - \; \langle bba^*, [[b,a],b] \rangle \langle ab^*, [a,b] \rangle \\
&= \big(\langle ab^*, [b,a] \rangle \langle b^*, b \rangle \; - \; \langle bb^*, [b,a] \rangle \langle a^*, b \rangle \big)(-1) \\
&\quad - \; \big(\langle bb^*, [b,a] \rangle \langle a^*, b \rangle \; - \; \langle ba^*, [b,a] \rangle \langle b^*, b \rangle \big)(1) \\
&= (1)(-1) - (1)(1) = -2.
\end{aligned}$$

An alternate method of computing $\langle -, - \rangle$ will follow from our work in Section 3 (see Proposition 3.12 and Example 3.14).

The map p_A to universal (associative) enveloping algebras is an injection by a simple corollary of the Poincaré-Birkhoff-Witt theorem. Thus Proposition 2.4 has the following corollary.

Corollary 2.6. *The map $\eta_A : (\mathbb{T}V)^* \to (\mathbb{L}V)^*$ is a surjection of coalgebras.*

Write $\big\langle \ker(\,]\cdot[\,) \big\rangle \subset (\mathbb{T}V)^*$ for the smallest coideal of $(\mathbb{T}V)^*$ containing $\ker(\,]\cdot[\,) \backslash V^*$. The following proposition is implied by various classical results; we will give a new, simple proof of it later. (Proposition 2.7 follows immediately from Proposition 3.15; Proposition 3.15 is a direct consequence of Proposition 4.15; Proposition 4.15 is simple to prove.)

Proposition 2.7. $\ker(\eta_A) = \big\langle \ker(\,] \cdot [\,) \big\rangle$. *In particular,* $\ker(\eta_A)$ *is a coideal of* $(\mathbb{T}V)^*$.

Corollary 2.8. $(\mathbb{L}V)^* \cong (\mathbb{T}V)^* / \ker(\eta_A)$ *as coalgebras.*

Remark 2.9. The idea that $(\mathbb{L}V)^* \cong (\mathbb{T}V)^* / \big\langle \ker(\,] \cdot [\,) \big\rangle$ is already present in the first section of [10] (and probably elsewhere in the literature as well), developed using Hopf algebra structures, dual to classical work of Quillen [7]. The idea that $(\mathbb{L}V)^* \cong (\mathbb{T}V)^* / \ker(\eta_A)$ (as vector spaces) is contained in the classical approach to Lie algebras via Lie (or Hall) polynomials – for example [9, §4.2].

It remains to describe $\ker(\eta_A)$ explicitly. Classically $\big\langle \ker(\,] \cdot [\,) \big\rangle$ is the vector subspace spanned by shuffles. The shuffle of two words is defined recursively by $\mathrm{Sh}(a, b) = ab + ba$ for a, b single letters and $\mathrm{Sh}(\omega, av) = \mathrm{Sh}(av, \omega) = \big(a\,\mathrm{Sh}(v, \omega) + \mathrm{Sh}(v, \omega)\,a\big)$ for v, ω generic words. Recall that the (associative)

Lyndon-Shirshov words in an ordered alaphabet [9, §5] [2] are the words which are lexicographically less than each of their cyclic permutations. By [8] the Lyndon-Shirshov words are a multiplicative basis for the shuffle monoid. Thus if we choose an ordered basis for V, then the Lyndon-Shirshov words in that basis are a vector space basis for $(\mathbb{T}V)^*/\ker(\eta_A)$. (Another basis for $(\mathbb{T}V)^*/\ker(\eta_A)$ is given in [15], using the configuration pairing.)

There are two ways to improve the presentation of $(\mathbb{L}V)^*$ given above. The first is to move away from associative algebras, since they are often not a convenient location for constructive proofs (see for example [3, Prop. 22.8] compared to [14, Lemma 2.16]). The second is to find a description of $\ker(\eta)$ not involving shuffles, since their span is rather complicated. For example, $a_1a_2a_3a_4a_5 - a_5a_4a_3a_2a_1$ is in this span, even though it is far from being a shuffle; neither is it immediately apparent how to write it as a sum of shuffles. In particular, using Corollary 2.8 in order to make a construction on $(\mathbb{L}V)^*$ involves making a construction on $(\mathbb{T}V)^*$ and then showing it is invariant under shuffles. The invariance step is difficult.

Other descriptions of the kernel of η_A follow from our later work in Sections 3 and 4, and are constructed directly from the map η_A using the configuration pairing rather than via the cobracket.

3. The Configuration Pairing with PreLie Colgebras

3.1. **PreLie algebras.**

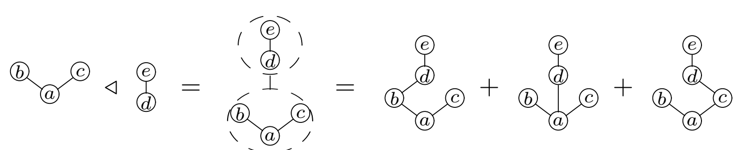
Definition 3.1. A preLie algebra [4] is $(P, \triangleleft)$ where $\triangleleft : P \otimes P \to P$ satisfies

$$(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) = (x \triangleleft z) \triangleleft y - x \triangleleft (z \triangleleft y).$$

The name "preLie" comes from the fact that $[x, y] = x \triangleleft y - y \triangleleft x$ is a Lie bracket. Note that all associative algebras are trivially preLie.

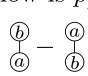
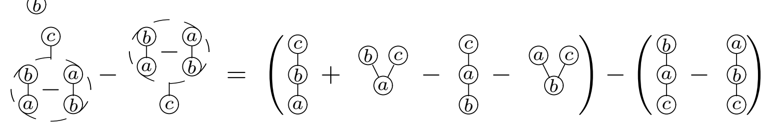
Free preLie algebras have a simple combinatorial model [1]. Given a vector space V, the free preLie algebra on V is the vector space of rooted (nonplanar) trees with vertices decorated by elements of V, modulo multilinearity. The algebra structure $x \triangleleft y$ is given on homogeneous elements by summing over all possible ways to connect the root of the rooted tree y to any one of the vertices of x. It is straightforward to show that this satisfies Definition 3.1. Write $\mathbb{P}V$ for the free preLie algebra on V, which we view as the vector space of multilinear vertex-labeled, rooted trees.

Example 3.2. Below we give the preLie operation $\triangleleft$ combining two rooted trees. We indicate the root of a tree by writing it as the unique lowest vertex.



A Lie algebra L has universal enveloping preLie algebra $U_PL = \mathbb{P}L/\sim$ where $[x, y] \sim x \triangleleft y - y \triangleleft x$. The preLie polynomial map $p_p : L \to U_PL$ is the composition $L \hookrightarrow \mathbb{P}L \twoheadrightarrow U_PL$. It follows from adjointness properties that $U_P\mathbb{L}V \cong \mathbb{P}V$. We are interested in the map $p_p : \mathbb{L}V \to \mathbb{P}V$ and its dual.

Example 3.3. Below is $p_p(\ell)$ for two simple Lie bracket expressions.

- $[a, b] \longmapsto$ 
- $[[a, b], c] \longmapsto$ 

3.2. **PreLie configuration pairing.** Write $(\mathbb{P}V)^*$ for the vector space dual of $\mathbb{P}V$. The remark about infinite dimensional coalgebra structures in the previous section still applies.

Definition 3.4. Define the vector space pairing $\langle -, - \rangle : (\mathbb{P}V)^* \otimes \mathbb{L}V \to k$ to be $\langle \phi,\, x\rangle = \phi\big(p_p(x)\big)$.

Let $\eta_p : (\mathbb{P}V)^* \to (\mathbb{L}V)^*$ be the associated map $\phi \mapsto \langle \phi,\, -\rangle$.

Remark 3.5. The map η_p is the dual of p_p as a map of vector spaces.

Given a chosen basis B of V, preLie polynomials are $p_p(\ell) = \sum_i c_i\, r_i$ where the r_i are rooted trees with vertices labeled from B. If r is a rooted tree with labels from B, then $\langle r^*,\, \ell\rangle$ is the coefficient of r in $p_p(\ell)$.

The induced coalgebra structure on $(\mathbb{P}V)^*$ is a sum, cutting each edge of a tree in turn to divide it into two trees, writing (root tree)$\otimes$(branch tree). As before, define the cobracket to be the anti-commutative twist $]r^*[\; = \sum\limits_{e\in \mathrm{E}(r)} (r_1^{\hat{e}})^* \otimes (r_2^{\hat{e}})^* - (r_2^{\hat{e}})^* \otimes (r_1^{\hat{e}})^*$, where $\sum_e$ is a sum over all edges of r and $r_1^{\hat{e}}, r_2^{\hat{e}}$ are the rooted trees obtained by removing edge e, numbered so that $r_1^{\hat{e}}$ is the subtree containing the root of r; the root of $r_2^{\hat{e}}$ is the vertex formerly incident to e.

The following are proven as in Section 2. [Corollary 3.7 requires knowing p_p is an injection, which we show in the appendix.]

Proposition 3.6. *If $\phi \in (\mathbb{P}V)^*$ then $\big\langle \phi,\, [x,y]\big\rangle \;=\; \sum_i \big\langle \alpha_i,\, x\big\rangle \big\langle \beta_i,\, y\big\rangle$, where $]\phi[\; = \sum_i \alpha_i \otimes \beta_i$.*

Corollary 3.7. *The map $\eta_p : (\mathbb{P}V)^* \to (\mathbb{L}V)^*$ is a surjection of coalgebras.*

There is an alternate, combinatorial definition of $\langle -, - \rangle$. Recall that Lie bracket expressions can be written as leaf-labeled, rooted, planar binary trees in the following manner. Define the tree associated to $v \in V \subset \mathbb{L}V$ to be $\overset{v}{|}$ (the rooted tree with a single leaf decorated by v), and recursively define the tree associated with a bracket $[\ell_1, \ell_2]$ to be ${}^{\tau_1}\!\!\curlyvee^{\tau_2}$, where τ_1, τ_2 are the trees associated to ℓ_1, ℓ_2. Call such trees Lie trees.

Remark 3.8. The combinatorial definition of the configuration pairing comes from the following two observations. Let $r^* \in (\mathbb{P}V)^*$ and $\ell \in \mathbb{L}V$ be homogeneous elements written in terms of a chosen basis B of V. Recursively applying Proposition 3.6 reduces $\langle r^*,\, \ell\rangle$ to a sum of products of $\langle b_i^*,\, b_j\rangle$ where the b_i are vertex labels of r and the b_j are leaf labels of ℓ. Thus, for $\langle r^*,\, \ell\rangle \neq 0$ there must be a label-preserving bijection between the vertices of r and the leaves of ℓ.

Futhermore, iterating the application of Proposition 3.6, the operation of cutting an edge of r corresponds to ungrafting subtrees from ℓ. Thus the edges of r and subtrees of ℓ should also be bijective. The subtrees of ℓ are enumerated by their base vertices, which are the internal vertices of ℓ.

Given two leaves a and b of a Lie tree, write $\mathrm{root}(a, b)$ for the internal vertex supporting the smallest subtree containing both leaf a and leaf b. [Alternately $\mathrm{root}(a, b)$ is the internal vertex closest to the root on the path from a to b. It is also the first common vertex of the paths from a and b to the root.]

Definition 3.9. Let R, T be unlabeled preLie and Lie trees. A bijection $\sigma : \mathrm{Vertices}(R) \xrightarrow{\cong} \mathrm{Leaves}(T)$ induces a map $\beta_\sigma : \mathrm{Edges}(R) \to \{\text{internal vertices of } T\}$ by setting $\beta_\sigma\left(\begin{smallmatrix} ⓑ \\ | \\ ⓐ \end{smallmatrix}\right) = \mathrm{root}(a, b)$.

The σ-configuration pairing of unlabeled preLie trees and Lie trees R and T is given by the following.

$$\langle R,\, T\rangle_\sigma = \begin{cases} \prod\limits_{e\in \mathrm{E}(R)} \mathrm{sgn}\big(\beta_\sigma(e)\big) & \text{if } \beta_\sigma \text{ is surjective} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\prod_e$ is a product over all edges of R and $\mathrm{sgn}\left(\beta_\sigma\left(\begin{smallmatrix} ⓑ \\ | \\ ⓐ \end{smallmatrix}\right)\right) = \pm 1$ depending on whether leaf $\sigma(a)$ is left or right of leaf $\sigma(b)$.[1]

[1]The placement of ⓐ below ⓑ is intended to indicate that ⓐ is the vertex closer to the root.

Example 3.10. Following is the map β_σ for two different isomorphisms σ of the vertices of a rooted preLie tree and leaves of a Lie tree. The different isomorphisms are indicated by numbering.

Above, $\mathrm{sgn}(\beta_{\sigma_1}(e_i)) = 1$ and $\mathrm{sgn}(\beta_{\sigma_2}(e_i)) = -1$, with pairings $\langle R, T\rangle_{\sigma_1} = 1$ and $\langle R, T\rangle_{\sigma_2} = 0$.

The σ-configuration pairing satisfies a property analogous to Proposition 3.6. Given an unlabeled preLie tree R, write $R_1^{\hat{e}}$ and $R_2^{\hat{e}}$ for the rooted trees obtained by removing the edge e from R, numbered so that the root of $R_1^{\hat{e}}$ is the root of R and the root of $R_2^{\hat{e}}$ is the vertex formerly incident to e. Also, given $\sigma : \mathrm{Vertices}(R) \xrightarrow{\cong} \mathrm{Leaves}(T)$ and a subtrees R' of R and T' of T, set

$$\big\langle R',\, T'\big\rangle_\sigma = \begin{cases} \big\langle R',\, T'\big\rangle_{\sigma|_{R'}} & \text{if } \sigma\big(\mathrm{Vertices}(R')\big) = \mathrm{Leaves}(T'), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.11. *With the notation above*

$$\big\langle R,\, [T_1,\, T_2]\big\rangle_\sigma \;=\; \sum_{e\in E(R)} \big\langle R_1^{\hat{e}},\, T_1\big\rangle_\sigma \big\langle R_2^{\hat{e}},\, T_2\big\rangle_\sigma \;-\; \big\langle R_2^{\hat{e}},\, T_1\big\rangle_\sigma \big\langle R_1^{\hat{e}},\, T_2\big\rangle_\sigma$$

where $\sum_e$ is a sum over the edges of R. Furthermore, at most one term in the above sum is nonzero.

Proof. Write $\sigma^{-1}T_i$ for the full subgraphs of R on the vertices $\sigma^{-1}\mathrm{Leaves}(T_i)$. It is enough to prove that

$$\big\langle R,\, [T_1, T_2]\big\rangle_\sigma = \begin{cases} \pm\big\langle \sigma^{-1}T_1,\, T_1\big\rangle_\sigma \big\langle \sigma^{-1}T_2,\, T_2\big\rangle_\sigma & \text{if } \sigma^{-1}T_i \text{ are connected,} \\ 0 & \text{otherwise} \end{cases}$$

where the $\pm$ sign above depends on whether the root of R is in $\sigma^{-1}T_1$ or in $\sigma^{-1}T_2$.

If the $\sigma^{-1}T_i$ are connected then Definition 2 breaks up as claimed by straightforward combinatorics. In this case $R_1^{\hat{e}}$ and $R_2^{\hat{e}}$ are $\sigma^{-1}T_1$ and $\sigma^{-1}T_2$.

If one of $\sigma^{-1}T_i$ is not connected, then β_σ will not be a surjection, because there will be less edges in $\sigma^{-1}T_i$ than internal vertices of T_i. In this case $\big\langle R,\, [T_1, T_2]\big\rangle_\sigma = 0$ by definition. □

Given $r \in \mathbb{P}V$ and $\ell \in \mathbb{L}V$, write $|r|$ and $|\ell|$ for the underlying unlabeled rooted preLie and Lie trees of r and ℓ. Also, write $l_r : \mathrm{Vertices}\big(|r|\big) \to S_r$ and $l_\ell : \mathrm{Leaves}\big(|\ell|\big) \to S_\ell$ for their labeling functions.

Proposition 3.12. *For homogeneous $r^* \in (\mathbb{P}V)^*$ and $\ell \in \mathbb{L}V$, the configuration pairing of Definition 3.4 is equal to the following.*

$$\big\langle r^*,\, \ell\big\rangle \;=\; \sum_{\sigma:\mathrm{V}(r)\xrightarrow{\cong}\mathrm{L}(\ell)} \left(\big\langle |r|,\, |\ell|\big\rangle_\sigma \prod_{v\in \mathrm{V}(r)} \Big\langle l_r(v)^*,\, l_\ell\big(\sigma(v)\big)\Big\rangle \right) \tag{3}$$

where $\sum_\sigma$ is a sum over all isomorphisms σ : Vertices$(r) \to$ Leaves(ℓ), $\prod_v$ is a product over all vertices of r, and $\langle l_r(v)^*,\, l_\ell(\sigma(v))\rangle$ is the canonical pairing of V^* and V. If there are no isomorphisms σ, then $\langle r^*,\, \ell\rangle = 0$.

Proof. Rather than attempt to make Remark 3.8 explicit, to prove the proposition it is enough to note that (3) matches Definition 3.4 when ℓ is a trivial Lie bracket and then apply strong induction on bracket length using Proposition 3.6 and Lemma 3.11. □

Corollary 3.13. *For $r^* \in (\mathbb{P}V)^*$ and $\ell \in \mathbb{L}V$ homogeneous, written in terms of a chosen basis B of V,*

$$\big\langle r^*,\, \ell\big\rangle \;=\; \sum_{\substack{\sigma:V(r)\xrightarrow{\cong}L(\ell) \\ \textit{label-preserving}}} \big\langle |r|, |\ell|\big\rangle_\sigma .$$

Corollary 3.13 combined with Proposition 3.18 gives an alternative to the recursive method of Proposition 2.4 (applied in Example 2.5) for the calculation of $\langle\psi, x\rangle$. The proof of the following example/application is left to the reader. (The following could also be proven using 2.4, but it is immediately obvious using 3.13 and i_p.)

Example 3.14. Suppose x is a right-normed bracket expression $x = [a_1, [a_2, [\ldots, [a_{n-1}, a_n]]]]$ and w is a word $w = w_1 \cdots w_n$. Then $\langle w^*, x\rangle$ is a signed count of the number of ways that $a_1 a_2 a_3 \cdots a_n$ can be written as $w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)} w_k$ where σ is a shuffle of $(1, \ldots, k-1)$ into $(n, \ldots, k+1)$ with sign $(-1)^{n-k}$.

In other words, reading w left-to-right should read x moving left-to-right skipping some letters and then should read the skipped letters right-to-left. The sign comes from the number of times you move right-to-left as in the examples below.

- $\langle abcdef^*,\ [a, [f, [b, [e, [c, d]]]]]\rangle = 1$ $\qquad [a,\ \ [f,\ \ [b,\ \ [e,\ \ [c,\ \ d]]]]]$
- $\langle abcdef^*,\ [f, [a, [e, [b, [d, c]]]]]\rangle = -1$ $\qquad [f,\ \ [a,\ \ [e,\ \ [b,\ \ [d,\ \ c]]]]]$
- $\langle abcdef^*,\ [f, [e, [a, [c, [b, d]]]]]\rangle = 0$ $\qquad [f,\ \ [e,\ \ [a,\ \ [c,\ \ [b,\ \ d]]]]]$
- $\langle abbab^*,\ [a, [b, [b, [b, a]]]]\rangle = -3$

$$[a,\ \ [b,\ \ [b,\ \ [b,\ \ a]]]] \ + \ [a,\ \ [b,\ \ [b,\ \ [b,\ \ a]]]] \ + \ [a,\ \ [b,\ \ [b,\ \ [b,\ \ a]]]]$$

There is a similar statement for left-normed bracket expressions.

3.3. **Lie coalgebras via the preLie configuration pairing.** Let $\big\langle \ker(\,]\cdot[\,)\big\rangle \subset (\mathbb{P}V)^*$ be the smallest coideal of $(\mathbb{P}V)^*$ containing $\ker(\,]\cdot[\,)\backslash V^*$. We postpone the proof of the following until after Proposition 4.15.

Proposition 3.15. $\ker(\eta_p) = \big\langle \ker(\,]\cdot[\,)\big\rangle$. *In particular,* $\ker(\eta_p)$ *is a coideal of* $(\mathbb{P}V)^*$.

Corollary 3.16. $(\mathbb{L}V)^* \cong (\mathbb{P}V)^*/\ker(\eta_p)$ *as coalgebras.*

Proposition 3.15 implies Proposition 2.7 (the analogous proposition for $(\mathbb{T}V)^*$) in the following manner.

Definition 3.17. Let $i_p : (\mathbb{T}V)^* \hookrightarrow (\mathbb{P}V)^*$ be the dual of the algebra homomorphism $q_p : \mathbb{P}V \twoheadrightarrow \mathbb{T}V$ given by $q_p\big(a_1 \triangleleft (a_2 \triangleleft \cdots \triangleleft (a_{n-1} \triangleleft a_n))\big) = a_1 a_2 \cdots a_n$ and $q_p(r) = 0$ for rooted trees r not of this form.

Recall that the rooted tree $a_1 \triangleleft (a_2 \triangleleft \cdots \triangleleft (a_{n-1} \triangleleft a_n))$ has a_1 at the root, a_2 above a_1, a_3 above a_2, etc. It is easy to check that i_p is a coalgebra homomorphism for both the standard and the cobracket coalgebra structures of $(\mathbb{T}V)^*$ and $(\mathbb{P}V)^*$.

Proposition 3.18. $\eta_A = \eta_p \circ i_p$.

Proof. The proposition is trivially true on elements of $V^* \subset (\mathbb{T}V)^*$, where i_p is merely the identity map. Using strong induction on word length, applying Propositions 2.4 and 3.6, we get the result for all homogeneous elements of $(\mathbb{T}V)^*$. This implies the proposition for all of $(\mathbb{T}V)^*$. $\square$

Remark 3.19. The previous proposition is the dual of the statement $p_A = q_p \circ p_p$.

Corollary 3.20. *Let* $\psi \in (\mathbb{T}V)^*$. *Then* $\psi \in \ker(\eta_A)$ *if and only if* $i_p(\psi) \in \ker(\eta_p)$.

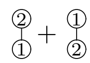
Proof of Proposition 2.7 assuming 3.15. Suppose that $\ker(\eta_p)$ is a coideal of $(\mathbb{P}V)^*$ and let $\psi \in \ker(\eta_A)$. By Corollary 3.20 $i_p(\psi) \in \ker(\eta_p)$, so $(i_p \otimes i_p)\big(\,]\psi[\,\big) = \,]i_p(\psi)[\, \in \big(\ker(\eta_p) \otimes (\mathbb{P}V)^*\big) \bigoplus \big((\mathbb{P}V)^* \otimes \ker(\eta_p)\big)$. Applying Corollary 3.20 again gives $]\psi[\, \in \big(\ker(\eta_A) \otimes (\mathbb{T}V)^*\big) \bigoplus \big((\mathbb{T}V)^* \otimes \ker(\eta_A)\big)$. This shows that $\ker(\eta_A)$ is a coideal of $(\mathbb{T}V)^*$.

For clarity, we write $\big\langle \ker(\,]\cdot[\,)\big\rangle_A$ and $\big\langle \ker(\,]\cdot[\,)\big\rangle_p$ for the coideal generated by $\ker(\,]\cdot[\,) \setminus V^*$ in $(\mathbb{T}V)^*$ and $(\mathbb{P}V)^*$ respectively. Suppose that $\ker(\eta_p) = \big\langle \ker(\,]\cdot[\,)\big\rangle_p$ and $\psi \in \ker(\eta_A)$. By the corollary $i_p(\psi) \in$

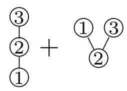
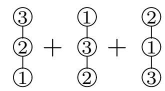
$\ker(\eta_p) = \big\langle \ker(\,]\cdot[\,) \big\rangle_p$. Since i_p is a coalgebra map, $i_p(\psi) \in \big\langle \ker(\,]\cdot[\,) \big\rangle_p$ if and only if $\psi \in \big\langle \ker(\,]\cdot[\,) \big\rangle_A$. Thus $\ker(\eta_A) \subset \big\langle \ker(\,]\cdot[\,) \big\rangle_A$. But $\ker(\eta_A)$ is a coideal, and by Proposition 2.4 it contains $\ker(\,]\cdot[\,) \setminus V^*$. Therefore $\ker(\eta_A) = \big\langle \ker(\,]\cdot[\,) \big\rangle_A$. $\square$

Now we describe $\ker(\eta_p)$. Define the weight of a rooted tree to be its number of vertices. From Example 3.3 we can read off $\ker(\eta_p)$ in low weights.

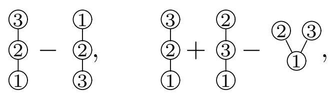
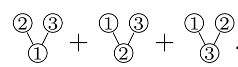
- In weight 2, $\ker(\eta_p)$ is given by replacing numbers by basis elements in the following expression.

(4) 

- In weight 3, $\ker(\eta_p)$ is spanned similarly by the following.

(5)  and 

Example 3.21. Other weight 3 expressions in $\ker(\eta_p)$ come from combining these. For example,

(6)  and 

Expression (4) is an anti-symmetry identity. The second expression of (5) is the Arnold identity. The first expression of (5) is merely (4) with the new vertex ③ added above ②. In general, $\ker(\eta_p)$ is local in the sense that grafting rooted trees onto expressions in the kernel yields new kernel expressions.

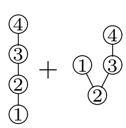
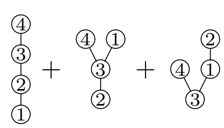
Definition 3.22. Given rooted trees R, T and a chosen vertex v of R, the grafting of T onto R at v is $(R\,{}_v\!\lhd T)$, the rooted tree given by adding an edge between the root of T and the vertex v of R.

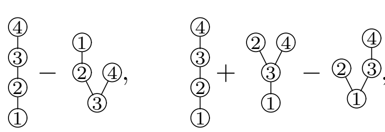
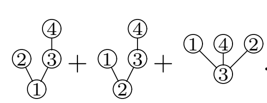
Call a labeled rooted tree simple if its set of labels is linearly independent.

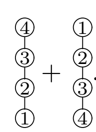
Proposition 3.23. *Let $r_1^* + \cdots + r_n^* \in \ker(\eta_p)$ with r_i weight m simple trees and let $t \in \mathbb{P}V$ be homogeneous.*
Given $v_i \in$ Vertices(r_i) all with the same label, $(r_1\,{}_{v_1}\!\lhd t)^* + \cdots + (r_n\,{}_{v_n}\!\lhd t)^* \in \ker(\eta_p)$.
If the roots of r_i all have the same label, then for any vertex v of t, $(t\,{}_v\!\lhd r_1)^* + \cdots + (t\,{}_v\!\lhd r_n)^* \in \ker(\eta_p)$.

Proof. From Definition 3.9 of the σ-configuration pairing $\big\langle (r_v\!\lhd t),\, \ell \big\rangle_\sigma = \begin{cases} \pm\langle r,\, \sigma(r)\rangle_\sigma \, \langle t,\, \sigma(t)\rangle_\sigma \\ 0 \end{cases}$ where $\sigma(r)$ is the subtree of ℓ containing the root and $\sigma(\text{Vertices}(r))$ and all of the paths from these leaves to the root. Apply Corollary 3.13. $\square$

Example 3.24. Grafting a new vertex ④ above ③ in the $\ker(\eta_p)$ elements of (5) and (6) yields the following weight 4 $\ker(\eta_p)$ elements.

(7)  and 

(8)  and 

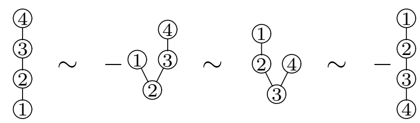
Combining the first expressions in (7) and (8) above yields the weight 4 anti-symmetry expression: 

Remark 3.25. The kernel of η_A does not have a local property such as this. For example, $ab-ba \in \ker(\eta_A)$; however, $abc-bac \notin \ker(\eta_A)$ and $abc-bca \notin \ker(\eta_A)$. We may attach c after b in ab; but we cannot attach c after b in ba without separating b and a.

Work in the next section implies the following.

Proposition 3.26. $\ker(\eta_p)$ *is generated by graftings onto anti-symmetry (4) and Arnold (5) expressions.*

Remark 3.27. The presence of roots in our trees makes the graftings of Proposition 3.23, and thus our understanding of $\ker(\eta_p)$, more complicated. However, from the point of view of $(\mathbb{L}V)^*$, roots should not play a central role. For example combining the first weight 4 kernel expressions of (7) and (8) it follows that modulo $\ker(\eta_p)$ the following are equivalent.



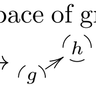
Grafting vertices onto the above relations yields similar relations shifting the root to arbitrary vertices of the weight n rooted tree $(a_1 \triangleleft (a_2 \triangleleft \cdots (a_{n-1} \triangleleft a_n)))$ modulo $\ker(\eta_p)$. Grafting onto these trees gives relations moving the root to arbitrary vertices of a generic preLie tree.

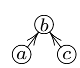
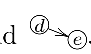
In the next section we replace rooted trees with directed graphs. This removes the artificial (from the point of view of $(\mathbb{L}V)^*$) distinction of the root element.

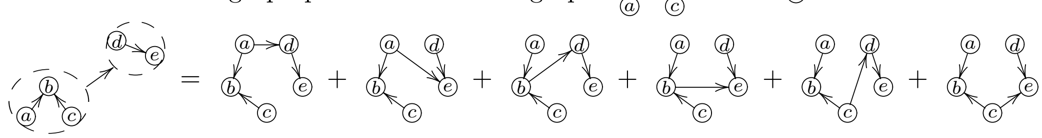
4. The Configuration Pairing with Graph Coalgebras

4.1. **Graph algebras.** We begin by describing the graph algebra map, which takes a vector space and makes an algebra. The graph algebra map is the free algebra map for a certain kind of algebra structure, but we will not elaborate on this point of view until the appendix. Instead we present graph algebras as a replacement for free preLie algebras in the sense that their dual contains preLie coalgebras in the same way that the preLie coalgebras contain associative coalgebras, and the kernel of the map η_G from graph coalgebras to lie coalgebras has a simple description.

For brevity, we say "graph" to mean directed, acyclic, connected, nonplanar graph.

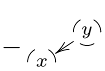
Definition 4.1. Let V be a vector space. Define $\mathbb{G}V$ to be the vector space of graphs with vertices labeled by elements of V, modulo multilinearity. The graph product $g \otimes h \longmapsto$  $\in \mathbb{G}V$ is the bilinear map defined on homogeneous elements as a sum over all of the ways of adding a directed edge from a vertex of g to a vertex of h.

Example 4.2. Below is the graph product of the two graphs  and .

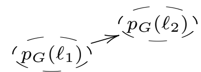
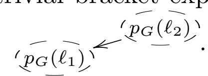


By a straightforward calculation, graph products satisfy Definition 3.1.

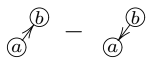
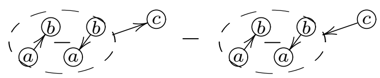
Proposition 4.3. $\mathbb{G}V$ *is a preLie algebra.*

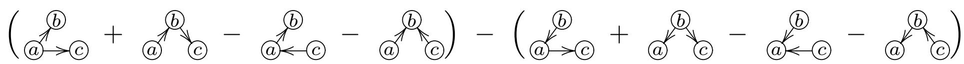
Corollary 4.4. *The bracket* $[x,y] =$  $-$  *makes* $\mathbb{G}V$ *a Lie algebra.*

Remark 4.5. We show in the appendix that $\mathbb{G}V$ has more structure than just that of a preLie algebra. Specifically it has extra, higher products which are not given by compositions of the binary product. In fact, preLie algebras are graph algebras whose only nontrivial higher products are those generated by the binary product.

Since $\mathbb{G}V$ is a Lie algebra, there is a unique map $p_G : \mathbb{L}V \to \mathbb{G}V$ sending trivial bracket expressions to trivial graphs. Defined recursively p_G is $p_G\big([\ell_1, \ell_2]\big) =$  $-$ . In the appendix, we construct p_G more generally via the universal enveloping graph algebra of a Lie algebra and we show that $p_G : L \to U_G L$ is an injection.

Example 4.6. Below is $p_G(\ell)$ for two simple Lie bracket expressions.

- $[a, b] \longmapsto$ 
- $[[a, b], c] \longmapsto$ 

$=$ 

4.2. **Graph configuration pairing.** As before, write $(\mathbb{G}V)^*$ for the vector space dual. The induced coalgebra structure on $(\mathbb{G}V)^*$ cuts a graph at all edges, writing (source graph) $\otimes$ (target graph). Define the cobracket to be the anti-commutative twist $]g^*[\; = \sum_{e \in E(g)} (g_1^{\hat{e}})^* \otimes (g_2^{\hat{e}})^* - (g_2^{\hat{e}})^* \otimes (g_1^{\hat{e}})^*$, where $\sum_e$ is a sum over the edges of g and $g_1^{\hat{e}}$, $g_2^{\hat{e}}$ are the graphs obtained by removing edge e which went from $g_1^{\hat{e}}$ to $g_2^{\hat{e}}$. We omit the proofs below which are identical to those of Section 3.

Definition 4.7. Define the vector space pairing $\langle -, - \rangle : (\mathbb{G}V)^* \otimes \mathbb{L}V \to k$ by $\langle \gamma, x \rangle = \gamma\big(p_G(x)\big)$.
Let $\eta_G : (\mathbb{G}V)^* \to (\mathbb{L}V)^*$ be the map $\gamma \mapsto \langle \gamma, - \rangle$.

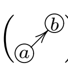
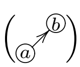
Remark 4.8. η_G is the dual of p_G as a map of vector spaces.

If V has chosen basis B, then the elements of $(\mathbb{G}V)^*$ are uniquely written as formal linear combinations of g^* where g are graphs with vertex labels from B. In this case, $\langle g^*, \ell \rangle$ calculates the coefficient of g in $p_G(\ell)$.

Proposition 4.9. *If $\gamma \in (\mathbb{G}V)^*$ then $\big\langle \gamma, [x, y] \big\rangle = \sum_i \langle \alpha_i, x \rangle \, \langle \beta_i, y \rangle$, where $]\gamma[\; = \sum_i \alpha_i \otimes \beta_i$.*

Corollary 4.10. *The map $\eta_G : (\mathbb{G}V)^* \to (\mathbb{L}V)^*$ is a surjection of coalgebras.*

Definition 4.11. Define β_σ and $\langle -, - \rangle_\sigma$ for graphs as in Definition 3.9:

$\beta_\sigma\Big($  $\Big) = \operatorname{root}(a, b)$ and $\operatorname{sgn}\Big(\beta_\sigma\Big($  $\Big)\Big) = \pm 1.$

$$\langle G, T \rangle_\sigma = \prod_{e \in \mathrm{E}(G)} \operatorname{sgn}\big(\beta_\sigma(e)\big) \qquad \text{if } \beta_\sigma \text{ is surjective.}$$

The following proposition connects to the configuration pairing of [13] and [14].

Proposition 4.12. *On homogeneous elements, the graph configuration pairing is given by the following.*

$$\big\langle g^*, \ell \big\rangle = \sum_{\sigma : V(g) \xrightarrow{\cong} L(\ell)} \left(\big\langle |g|, |\ell| \big\rangle_\sigma \prod_{v \in V(g)} \Big\langle l_g(v)^*, l_\ell\big(\sigma(v)\big) \Big\rangle \right)$$

Corollary 4.13. *For $r^* \in (\mathbb{G}V)^*$ and $\ell \in \mathbb{L}V$ homogeneous, written in terms of a chosen basis B of V,*

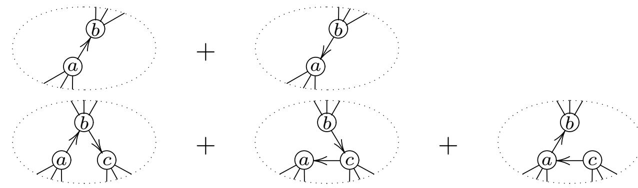
$$\big\langle g^*, \ell \big\rangle \; = \sum_{\substack{\sigma : V(g) \xrightarrow{\cong} L(\ell) \\ \textit{label-preserving}}} \big\langle |g|, |\ell| \big\rangle_\sigma .$$

4.3. **Lie coalgebras via the graph configuration pairing.** We begin with the analog of Proposition 3.26. The following proposition appears in [12] as Proposition 1.6 and Theorem 1.8. Due to its importance and simplicity, we include an outline of the proof.

Proposition 4.14. $\ker(\eta_G)$ *is generated by local arrow-reversing and Arnold expressions of graphs:*

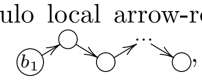
(arrow-reversing)

(Arnold)



where ⓐ, ⓑ, and ⓒ are vertices in a graph which is fixed outside of the indicated area.

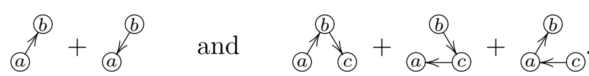
Proof. By a computation using Corollary 4.13, the above expressions are in $\ker(\eta_G)$.

To show that these span the entire kernel, note that modulo local arrow-reversing and Arnold, all graphs are linear combinations of "long" graphs, of the form , and recall that modulo anti-symmetry and Jacobi all Lie brackets are linear combinations of right-normed Lie bracket expressions $[b_1, [-, \cdots [-, -]]]$ where b_1 is a fixed generator. A short computation using Corollary 4.13 shows that the "long" graphs above pair perfectly under $\langle -, - \rangle$ with right-normed Lie brackets. Since right-normed Lie brackets span $\mathbb{L}V$, arrow-reversing and Arnold expressions must span $\ker(\eta_G)$. $\square$

The following is Propositions 3.7 and 3.18 of [13]. Its corollary is the main tool of [13].

Proposition 4.15. $\ker(\eta_G) = \big\langle \ker(\,] \cdot [\,) \big\rangle$. *In particular,* $\ker(\eta_G)$ *is a coideal of* $(\mathbb{G}V)^*$.

Proof. By Proposition 4.14, to show that $\ker(\eta_G)$ is a coideal, it is enough to check that the cobracket of the below arrow reversing and Arnold expressions land in $\big(\ker(\eta_G) \otimes (\mathbb{G}V)^*\big) \bigoplus \big((\mathbb{G}V)^* \otimes \ker(\eta_G)\big)$.



Similarly, to show that $\ker(\eta_G) \subset \big\langle \ker(\,] \cdot [\,) \big\rangle$ it is enough to check that the above expressions are in $\big\langle \ker(\,] \cdot [\,) \big\rangle$. $\square$

Corollary 4.16. $(\mathbb{L}V)^* \cong (\mathbb{G}V)^* / \ker(\eta_G)$ *as coalgebras.*

Proposition 3.15 follows from Proposition 4.15 in the same manner as Proposition 2.7 in Section 3.3.

Definition 4.17. A graph G is rooted if it has a vertex v such that every edge of G points away from v. In this case, call v the root of the graph G.

Define $q_G : \mathbb{G}V \twoheadrightarrow \mathbb{P}V$ to be the algebra homomorphism converting rooted graphs to rooted trees by forgetting edge directions (but remembering the root) and killing all non-rooted graphs.

Let $i_G : (\mathbb{P}V)^* \hookrightarrow (\mathbb{G}V)^*$ be the dual of q_G as a vector space map.

On homogeneous elements, $i_G(r^*) = g^*$ where g is the graph obtained by orienting each edge of the vertex-labeled, rooted tree r to point away from the root. It is clear that i_G is a coalgebra homomorphism for both the standard and cobracket coalgebra structures on $(TR)^*$ and $(TG)^*$.

Proposition 4.18. $\eta_p = \eta_G \circ i_G$.

Remark 4.19. Proposition 4.18 is dual to the statement $p_p = q_G \circ p_G$.

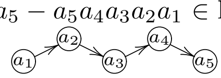
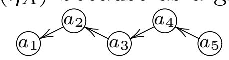
Corollary 4.20. *Let* $\phi \in (\mathbb{P}V)^*$. *Then* $\phi \in \ker(\eta_p)$ *if and only if* $i_p(\phi) \in \ker(\eta_G)$.

Proof of Proposition 3.15 assuming 4.15. This is identical to the corresponding proof in Section 3. $\square$

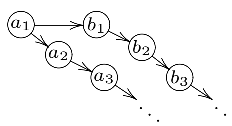
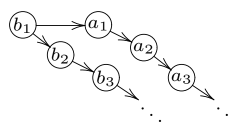
Proposition 3.26 follows from Proposition 4.14 using i_G similarly. Combining Propositions 4.18 and 3.18, we have the following corollaries.

Corollary 4.21. $\eta_A = \eta_G \circ i_G \circ i_p$.

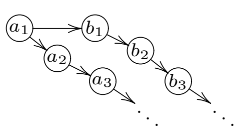
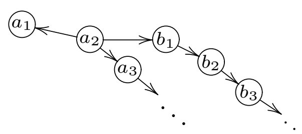
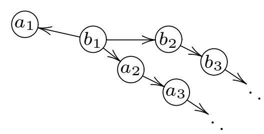
Corollary 4.22. *Let* $\psi \in (\mathbb{T}V)^*$. *Then* $\psi \in \ker(\eta_A)$ *if and only if* $(i_G \circ i_p)(\psi) \in \ker(\eta_G)$.

Example 4.23. Applying Corollary 4.22, parts of the $\ker(\eta_A)$ which were difficult to detect using shuffles are now obvious. For example, $a_1a_2a_3a_4a_5 - a_5a_4a_3a_2a_1 \in \ker(\eta_A)$ because as a graph coalgebra expression, $(i_G \circ i_r)(a_1a_2a_3a_4a_5 - a_5a_4a_3a_2a_1) =$  $-$  is just four applications of arrow-reversing.

Example 4.24. We get the shuffles inside of $\ker(\eta_A)$ via Corollary 4.22 in the following manner. Begin with the arrow-reversing expression

(9)  $+$ 

Note that the Arnold identity implies the following.

(10)  $\sim -$  $-$ 

Reversing the arrows to a_1 on the right-hand side above changes each sign. Iterating (10) beginning with the first term of (9) yields all shuffles of $(a_1a_2\cdots)$ into $(b_1b_2\cdots)$ with first letter a_1. Iterating (10) beginning with the second term of (9) yields all shuffles with first letter b_1.

Remark 4.25. Corollary 4.22 is the main component of [15] computing new bases for free Lie algebras. The computation consists of a series of combinatorial moves as in the previous example.

Appendix A. Operad Structures

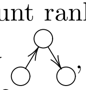
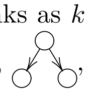
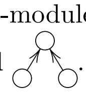
Operads are objects which encode algebra structures. On the set-level, they consist of an element for every possible way of combining elements using the algebra structure, along with composition maps expressing some combinations as compositions of others. More formally, a (unital, symmetric) operad $\mathcal{O}$ in the symmetric monoidal category of k-vector spaces is a symmetric sequence of vector spaces, $\{\mathcal{O}(n)\}_{n\geq 0}$ where each $\mathcal{O}(n)$ has Σ_n-action, equipped with a unit $k \to \mathcal{O}(0)$ and and equivariant composition maps, $\mathcal{O}(n) \bigotimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \mapsto \mathcal{O}(\sum_i k_i)$, satisfying standard unital and associativity axioms. The composition map tells which $(\sum_i k_i)$-ary operation is given by combining $k_1,\ldots,k_n$-ary operations together via an n-ary operation. The symmetric group action accounts for plugging elements into an n-ary operation in different orders. Below we use $\circ_i$ operations to define operad structure. These are maps $\mathcal{O}(n) \otimes \mathcal{O}(m) \mapsto \mathcal{O}(m+n-1)$ which plug an m-ary operation into an n-ary operation at position i.

- $\mathcal{A}s$. The associative operad is given by $\mathcal{A}s(n) \cong k[\Sigma_n]$ the regular representations of the symmetric groups. Composition is given by wreath product.
- $\mathcal{L}ie$. The Lie operad has $\mathcal{L}ie(n)$ given by the k-vector space generated by formal length n bracket expressions of the elements $a_1, \ldots, a_n$. This is isomorphic to the k-vector space of rooted binary planar trees whose leaf set is $[n] = \{1, \ldots, n\}$ (with Σ_n permuting $[n]$) modulo anti-symmetry and Jacobi identites of trees.

- $\mathcal{preLie}$. [1] The preLie operad is isomorphic to the operad of rooted trees. $\mathcal{preLie}(n)$ is the k-vector space of rooted trees with vertex set $[n]$. The operad structure of $\mathcal{preLie}(n)$ comes from the following $\circ_i$ operation. Direct the edges of a rooted tree to point away from the root. $R \circ_i T$ is given by replacing vertex i of R with the tree T. The incoming edge to i (if i is not the root of R) connects to the root of T, and we sum over all ways that the outgoing edges of i can be assigned source vertices in T.
- $\mathcal{Gr}$. The graph operad has $\mathcal{Gr}(n)$ given by the k-vector space of directed, acyclic graphs with vertex set $[n]$. The operad structure on $\mathcal{Gr}$ comes from the following $\circ_i$ operation. $G \circ_i H$ is given by replacing vertex i of G by the graph H, summing over all ways that edges with source or target vertex i can be assigned a new source or target vertex in H.

Remark A.1. Write $\mathcal{O}^\vee$ for the arity-wise dual of $\mathcal{O}$: i.e. $\mathcal{O}^\vee(n) = \mathcal{O}(n)^*$. If $\mathcal{O}$ is an (arity-wise finitely generated) operad, then $\mathcal{O}^\vee$ is a cooperad. The dual cooperad structure of $\mathcal{preLie}^\vee$ acts by quotienting subtrees to vertices. The dual cooperad structure of $\mathcal{Gr}^\vee$ acts by quotienting subgraphs to vertices, as described in [13].

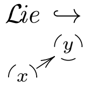
Proposition A.2. *$\mathcal{Gr}$ is not a quadratic operad* [5].

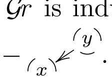
Proof. Count ranks as $k[\Sigma_n]$-modules. $\mathcal{Gr}(2)$ has rank 1 as a $k[\Sigma_2]$-module. $\mathcal{Gr}(3)$ is spanned as a $k[\Sigma_3]$-module by , , and . However, a quadratic operad with $\mathcal{O}(2)$ of rank 1 cannot have $\mathcal{O}(3)$ of rank > 2. $\square$

There are quotient maps of operads $\mathcal{Gr} \twoheadrightarrow \mathcal{preLie} \twoheadrightarrow \mathcal{As}$ defined as follows.

- The map $Q_p : \mathcal{preLie} \twoheadrightarrow \mathcal{As}$ is induced by the functor which views an associative algebra as a preLie algebra. Q_p takes rooted trees which are bivalent at all but two vertices to the permutation encoded by the vertices from the root to the leaf, and quotients all trees with a vertex n-valent, $n > 2$.
- The map $Q_G : \mathcal{Gr} \twoheadrightarrow \mathcal{preLie}$ is induced by the functor which views a preLie algebra as a graph algebra (by viewing an operation encoded by a rooted tree as an operation encoded by a rooted directed graph). Q_G takes rooted graphs to rooted trees, and quotients non-rooted graphs. [The interested reader may check that this commutes with $\circ_i$ operations.]

There are inclusion maps of operads $\mathcal{Lie} \hookrightarrow \mathcal{As}$, $\mathcal{Lie} \hookrightarrow \mathcal{preLie}$, $\mathcal{Lie} \hookrightarrow \mathcal{Gr}$ defined as follows.

- The map $U_A : \mathcal{Lie} \hookrightarrow \mathcal{As}$ is induced by the map viewing an associative algebra as a Lie algebra with bracket $[x, y] = xy - yx$. This map is an injection by Poincaré-Birkhoff-Witt.
- The map $U_p : \mathcal{Lie} \hookrightarrow \mathcal{preLie}$ is induced by the map viewing a preLie algebra as a Lie algebra with bracket $[x, y] = x \triangleleft y - y \triangleleft x$. From definitions, it follows that $U_A = Q_p U_p$. Since U_A is an injection, so is U_P.
- The map $U_G : \mathcal{Lie} \hookrightarrow \mathcal{Gr}$ is induced by the map viewing a graph algebra as a Lie algebra with bracket $[x, y] =$ 

$-$ 

. From definitions, it follows that $U_A = Q_p Q_G U_G$. Since U_A is an injection, so is U_G.

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